Towards NPT bound entanglement: computational complexity and field extensions

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Open problem 1: do there exist NPT bound entangled states?

Consider a bipartite quantum state $\rho \in \mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2}$.
Def. partial transpose: $\rho \mapsto (\mathrm{id}_{d_1} \otimes T_{d_2})\rho = \rho^{T_2}$.

Example: $\frac{1}{2}\begin{pmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}$

$\geqslant 0 \qquad\qquad \ngeqslant 0$

- PPT: Positive Partial Transpose $\rho^{T2} \geqslant 0$
- NPT: Negative Partial Transpose $\rho^{T2} \ngeqslant 0$

NPT $\implies$ entangled state

Alice & Bob

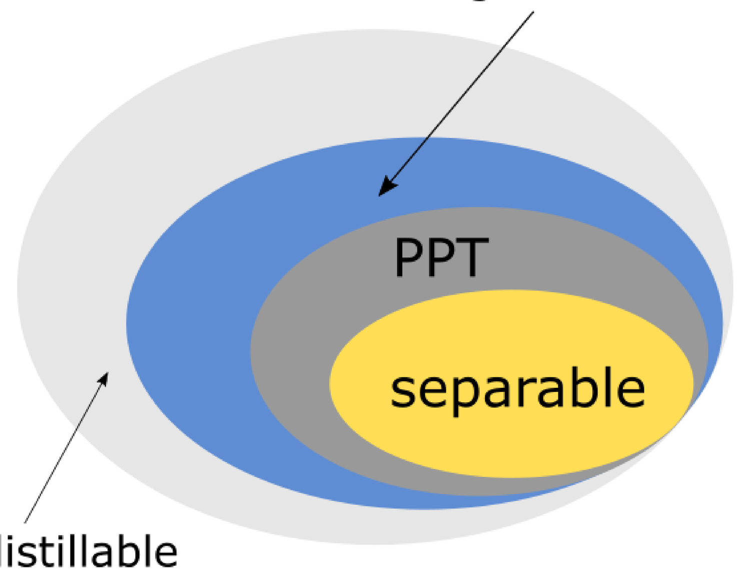
Goal: share $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
Tools: Local Operations & Classical Communication (LOCC)
Reality: ρ (noisy, mixed, ...)
Entanglement distillation:

n copies $\rho \ldots \rho$ $\xrightarrow{\text{LOCC}}$ $|\phi^+\rangle$

Is this possible?

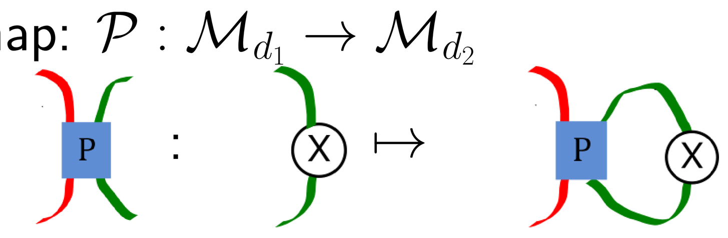
- yes: ρ distillable
- no: ρ bound entangled

PPT $\implies$ un-distillable

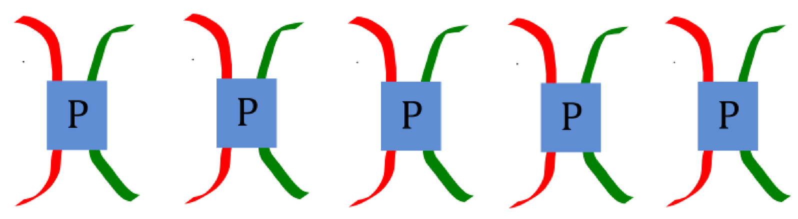


Open problem 2: Do there exist non-trivial tensor-stable postive maps?

Linear map: $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$

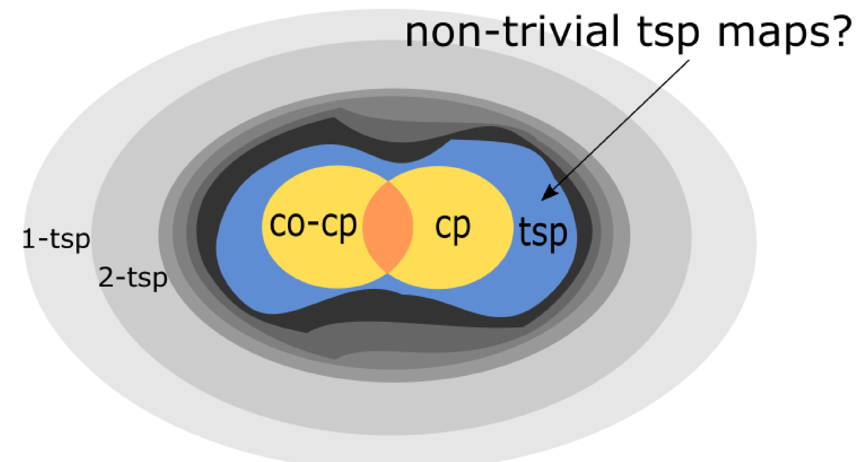


- Positive ($\mathcal{P} \succcurlyeq 0$) : if $X \geqslant 0$ then $\mathcal{P}(X) \geqslant 0$.
- Completely positive (cp): $\mathrm{id}_d \otimes \mathcal{P} \succcurlyeq 0$ for all d.
- Completely co-positive (co-cp): $\mathcal{P} = T \circ \mathcal{S}$, with $\mathcal{S}$ cp.



- n-tensor-stable positive : $\mathcal{P}^{\otimes n} \succcurlyeq 0$ for some $n \in \mathbb{N}$
- tensor-stable positive (tsp) : $\mathcal{P}^{\otimes n} \succcurlyeq 0$ for all $n \in \mathbb{N}$

'trivial' tsp maps: cp $\cup$ co-cp



Theorem 1: connection NPT bound entanglement and tsp (Mu16)

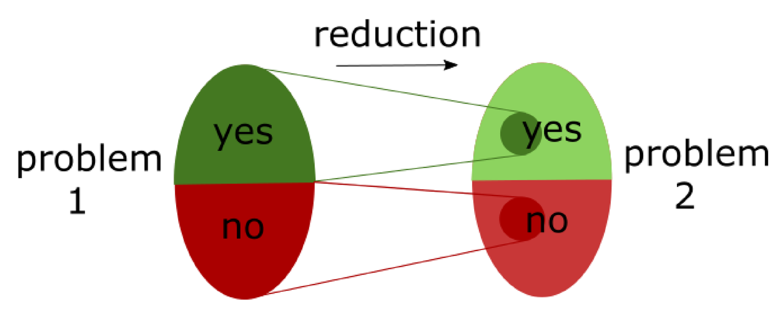
If there exists a non-trivial tensor-stable positive map $\mathcal{P} : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}$, then there exist NPT bound-entangled states in $\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_1}$ as well as in $\mathcal{M}_{d_2} \otimes \mathcal{M}_{d_2}$

Line 1

Line 2

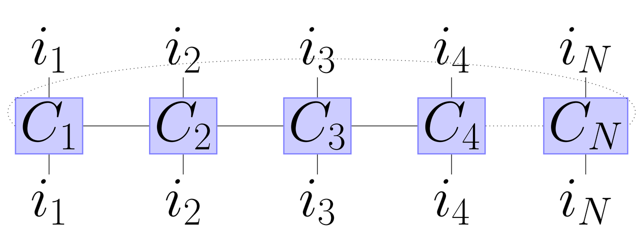
Computational complexity: undecidability

A decision problem is undecidable if there cannot exist an algorithm that gives the correct answer (yes/no) to every input.
Prove via reduction from another undecidable problem:



Problem MPO: undecidable (De16)

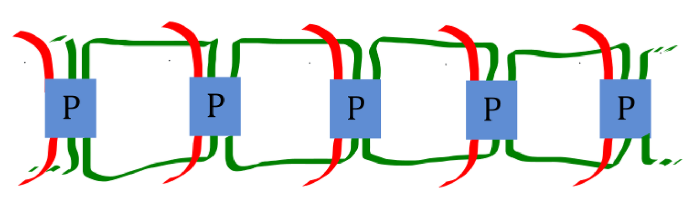
Given a Matrix Product Operator defined by C_i, is $\tau_N(C) \geqslant 0$ for all N?



Reduction ✔

Problem Bell pairs: undecidable

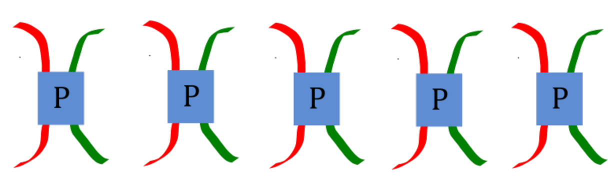
Given d and a linear map $\mathcal{P} : \mathcal{M}_d \to \mathcal{M}_d$, is $\mathcal{P}^{\otimes n}(\text{Bell pairs}) \geqslant 0$ for all n?



Reduction ✗

Problem TSP

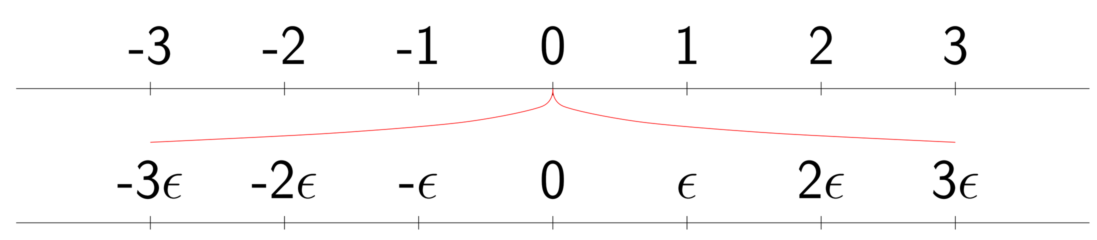
Given d and a linear map $\mathcal{P} : \mathcal{M}_d \to \mathcal{M}_d$, is $\mathcal{P}^{\otimes n} \succcurlyeq 0$ for all n?



Proving undecidability of Problem TSP would imply existence of NPT bound entangled states.

Field extensions: the hyperreals

The hyperreals ${}^*\mathbb{R}$ have additional infinitesimal elements ϵ.



Positive infinitesimal elements $\epsilon > 0 \in {}^*\mathbb{R}$ are smaller then all $r \in \mathbb{R}$. The hypercomplex ${}^*\mathbb{C}$ are defined as usual: ${}^*\mathbb{C} = {}^*\mathbb{R} + i{}^*\mathbb{R}$.

Theorem 2: Non-trivial tsp

There exist non-trivial tensor-stable positive maps $\mathcal{P} : \mathcal{M}_{d_1}({}^*\mathbb{C}) \to \mathcal{M}_{d_2}({}^*\mathbb{C})$ on the hypercomplex.

When transferred back to $\mathbb{C}$ however, the maps are trivial again.
Theorem 1 still holds on ${}^*\mathbb{C}$. Therefore:

Theorem 3: NPT bound entanglement

There exist NPT bound entangled states on ${}^*\mathbb{C}$.

These results can not (yet) be interpreted in a physical way.

Outlook and ongoing work

1. Explore other undecidable problems and provide a reduction to Probem TSP.
2. Prove existence of non-trivial tsp maps and NPT bound entanglement in an infinite dimensional Hilbert space.

(Mu16) A. Müller-Hermes, D.Reeb, and M. M. Wolf, Positivity of linear maps under tensor powers. J. Math. Phys. 57 (2016).
(De16) G. de las Cuevas, T. S. Cubitt, J. I. Cirac, M. M. Wolf, and D. Pérez-García, Fundamental limitations in the purifications of tensor networks. J. Math. Phys. 57 (2016).